International Journal of Theoretical Physics, Vol. 30, No. 12, 1991

Natural Gauges for Classical Charged Particles

R. G. Beil[1]

Received March 7, 1991

Alternate gauge conditions for classical charged particles are discussed. These gauges are suggested by unified descriptions of electromagnetism derived from general relativistic metrics which are velocity-dependent, Finsler metrics. In each of the examples it is shown that a particular gauge choice produces a potential which is related to the particle velocity and which identically satisfies the Lorentz charged particle equation. The results are related to Fermi–Walker propagation and are demonstrated to be consistent with the dynamics of both point and extended particles.

A continuing point for discussion in electrodynamics is the fact that the electromagnetic field $F_{\mu\nu} = \partial A_\nu/\partial x^\mu - \partial A_\mu/\partial x^\nu$ is not changed by a gauge transformation $A'_\mu = A_\mu + \partial\Lambda/\partial x^\mu$. This means that one is free to select a gauge by some limiting condition on the potential A_μ. A standard choice is the Lorentz gauge $\partial A^\mu/\partial x^\mu = 0$, which is simple and covariant. There are several other possibilities, however.

Of course, in the realm of quantum field theories the gauge uncertainty proliferates and there is a multitude of possible choices. A good summary of these is given in the article by Leibbrandt (1987).

The Lorentz gauge works fine for free fields and is also usually convenient when charge is present. It is satisfied by the Lienard–Wiechert potential for a point charge. But there has never been a compelling reason for the application of the Lorentz gauge to classical charged particles. As a matter of fact, in a recent theory of *extended* charged particles (Beil, 1989*a*) the Lorentz gauge is explicitly not satisfied by the self-potential of the particle except in the limit of large distance from the charge center.

One of the most notable examples of a possible alternative to the Lorentz gauge appears in the work of Dirac (1951). He first proposed the nonlinear condition $A_\mu A^\mu = K^2$ with K some universal constant. This was

[1]313 S. Washington, Marshall, Texas 75670.

0020-7748/91/1200-1663$06.50/0

equivalent to A_μ being proportional to the particle velocity v_μ. This was soon recognized by Gabor as being much too restrictive and in a second paper Dirac (1952) generalized his condition to one where A_μ is proportional to v_μ plus another term which he related to the vorticity of an electron stream.

Dirac's idea never gained much acceptance, but it has been recognized even recently (Parrott, 1987) as an elegant and potentially significant approach to the theory of charged particles. The theory suggests a connection between the electromagnetic field and the field of a continuous fluid.

Actually the idea of the potential being equal to a velocity term plus a second gaugelike term is in line with standard electromagnetic theory. The equation

$$A_\mu = -\frac{mc}{e} v_\mu + \frac{c}{e} \frac{\partial S}{\partial x^\mu} \tag{1}$$

is the usual relation between the potential and the action S. This is equivalent through a Hamilton–Jacobi analysis to the Lorentz equation for a charge particle in an external field (Rohrlich, 1965). Equation (1) is general and implies no gauge restriction on A_μ. However, (1) can also be considered to imply the form of a gauge transformation and can be used to specify a new potential, as will be demonstrated.

In recent years several alternate gauge transformations which are similar to (1) have been discussed.

One example, comparable to Dirac (1952), is given by Schweizer (1990). The gauge transformation is exactly (1) with a new potential

$$B_\mu = A_\mu - \frac{c}{e} \frac{\partial S}{\partial x^\mu} = -\frac{mc}{e} v_\mu \tag{2}$$

When this B_μ is inserted in the Lorentz equation

$$a^\mu = \eta^{\mu\nu} \frac{e}{mc} F_{\nu\alpha} v^\alpha = \eta^{\mu\nu} \frac{e}{mc} \left(\frac{\partial B_\alpha}{\partial x^\nu} - \frac{\partial B_\nu}{\partial x^\mu} \right) v^\alpha = \eta^{\mu\nu} \left(\frac{\partial v_\nu}{\partial x^\alpha} - \frac{\partial v_\alpha}{\partial x^\nu} \right) v^\alpha \tag{3}$$

it is not hard to see that the equation is identically satisfied. For this reason any potential B_μ which leads to (3) will be called a "natural" potential and the corresponding gauge a "natural" gauge.

Of course, here the action S comes from a Lagrangian

$$\mathscr{L} = (\eta_{\mu\nu} v^\mu v^\nu)^{1/2} + (e/mc)\eta_{\mu\nu} A^\mu v^\nu \tag{4}$$

which itself has Euler–Lagrange equations which are the Lorentz equation.

A second set of examples of natural gauges is related to a Lagrangian of the form

$$\mathscr{L}=[(\eta_{\mu\nu}+kB_\mu B_\nu)v^\mu v^\nu]^{1/2} \tag{5}$$

This Lagrangian is directly computed from a metric $g_{\mu\nu}=\eta_{\mu\nu}+kB_\mu B_\nu$. The constant k can be related to the gravitational constant (Beil, 1987). This Lagrangian can describe the motion of classical charged particles as well as (4). The Lagrangian has the form of a Finsler metric function, which allows the whole toolbox of Finsler geometric theory to be applied (Beil, 1989*b*). [Equation (4) is also a Finsler metric function of the Randers type.]

It was shown in the work of Fontaine and Amiot (1983) and also in Beil (1987) that the equation of motion for (5), if $B_\alpha v^\alpha=\text{const}$, is

$$a^\mu+k\eta^{\mu\nu}\left(\frac{\partial B_\nu}{\partial x^\alpha}-\frac{\partial B_\alpha}{\partial x^\nu}\right)(B_\beta v^\beta)v^\alpha=0 \tag{6}$$

If, for example,

$$B_\mu=A_\mu+\frac{\partial\Lambda}{\partial x^\mu} \tag{7}$$

where Λ is some scalar function and

$$B_\alpha v^\alpha=e/mck \tag{8}$$

then (6) is the Lorentz equation.

Fontaine and Amiot (1983) make a particular choice for Λ:

$$\Lambda=-\frac{c}{e}S+\left(\frac{e}{mck}+\frac{mc^3}{e}\right)\phi \tag{9}$$

The scalar function ϕ satisfies $(\partial\phi/\partial x^\mu)v^\mu=1$. It can be identified as an integral of the proper time along the trajectory of the particle, so that $\partial\phi/\partial x^\mu=\partial\tau/\partial x^\mu$.

One has, recalling (1),

$$B_\mu=-\frac{mc}{e}v_\mu+\left(\frac{e}{mck}+\frac{mc^3}{e}\right)\frac{\partial\tau}{\partial x^\mu} \tag{10}$$

as the expression for the gauge-transformed potential function. This is another natural potential, since it is related to (2) by a gauge.

In a subsequent development (Beil, 1987) the gravitational field equations for the metric (5) were derived and it was shown that k is related to the gravitational constant κ by $k=4\kappa/c^4$. Thus, a unified approach to electromagnetism and gravitation similar to Kaluza–Klein theories was obtained. A recent paper by Ferrari *et al.* (1989) also deals with the relation between Kaluza–Klein theories and this type of gauge condition.

A third class of natural gauges can be derived from a Lagrangian

$$\mathscr{L}=[(\eta_{\mu\nu}-kB_\mu B_\nu)v^\mu v^\nu]^{1/2} \tag{11}$$

This differs from (5) only in the sign. The equation of motion (for $B_\alpha v^\alpha=$ const) is

$$a^\mu-k\eta^{\mu\nu}\left(\frac{\partial B_\nu}{\partial x^\alpha}-\frac{\partial B_\alpha}{\partial x^\nu}\right)(B_\beta v^\beta)v^\alpha=0 \tag{12}$$

The sign difference is significant because it allows the use of new gauge conditions like

$$B_\mu=-\frac{1}{k^{1/2}c}v_\mu=\frac{e}{k^{1/2}c^2m}\left(A_\mu-\frac{c}{e}\frac{\partial S}{\partial x^\mu}\right) \tag{13}$$

This is compatible with (1) and is simpler than (10). It is easy to see that when (13) is used in (12) the Lorentz equation as well as the natural gauge result.

The consequences of the Lagrangian (11) remain to be explored. It should be mentioned, though, that the metric in this case can be written as

$$g_{\mu\nu}=\eta_{\mu\nu}-c^{-2}v_\mu v_\nu \tag{14}$$

which is similar to metrics studied by Synge (1971) and by Kawaguchi and Miron (1989) in generalized Lagrangian theory.

Each of the examples given above shares a general physical interpretation: For a particle in an external field there is a natural gauge in which the rate of change of the particle velocity can be related directly to the external field tensor,

$$F_{\mu\nu}=\frac{mc}{e}\left(\frac{\partial v_\mu}{\partial x^\nu}-\frac{\partial v_\nu}{\partial x^\mu}\right) \tag{15}$$

It is remarkable that this has exactly the form of the London equations of superconductivity (Ferrari *et al.*, 1989). A key difference, however, is that $F_{\mu\nu}$ is the external field in which the particle finds itself and not a self-field, so that there is no trapping of the field. It is not unreasonable, though, that since a free particle is, in some sense, in an environment with no resistance, an equation from superconductivity might apply.

As pointed out by Schweizer (1990), (15) means that for any potential A_μ one can find a gauge transformation to a potential B_μ which is tangent to the particle trajectories at all points.

A further physical insight, due to Konopinski (1978), is that the vector potential A_μ describes a "store" of field energy-momentum (or change of energy-momentum) which is contributing to the motion of the charge. Thus, the gauge transformation to B_μ only involves an additive term (a vector) which is similar to the additive constant associated with potential energy. So the vector potential (or more precisely the change in the vector potential) is physically measurable. This throws some light on the Bohm–Aharonov discussions. A theory of the Bohm–Aharonov effect which directly relates to this is given by Apsel (1979). This is also investigated by Ferrari *et al.* (1989).

It should be brought out that in a sense the above analysis implies no gauge condition at all, since no limitation on the original potential A_μ is established. In each of the above examples (1) still holds. What is specified each time is the particular gauge transformation which generates a new potential B_μ. The gauge condition then fixes B_μ.

It must be emphasized, however, that it is the external field which determines the motion of the particle and not vice versa. The gauge condition reflects the fact that the potential B_μ can be found such that the particle velocity is proportional to this potential.

On the other hand, for a different physical situation involving a fluid composed of charged particles an interpretation which realizes Dirac's original goal of relating the potential to the velocity of a fluid flow is made feasible. This development is explored by Ferrari *et al.* (1989).

In a different vein, it is interesting that at least three different Lagrangians, each with an associated gauge, all lead to the Lorentz equation as well as a natural potential. The theories begin to differ, though, when curvatures and gravitational field equations are computed. They are also quite different in Finsler geometry. It remains to be determined if this will lead to a clear preference among the three, or perhaps still other forms for the Lagrangian.

Some further developments can be made by looking at (15) in the coordinate system of the particle. In this system $R^\mu = x^\mu - z^\mu$ is the null vector from the center of the particle z^μ to a field point x^μ. The retarded distance from the particle center is $\rho = R_\mu v^\mu / c$. The proper time coordinate τ is identified as a field $\tau(x^\mu)$ with $\partial\tau/\partial x^\mu = R_\mu/\rho c$. One has

$$\frac{\partial v_\mu}{\partial x^\nu} = \frac{dv_\mu}{d\tau}\frac{\partial\tau}{\partial x^\nu} = \frac{a_\mu R_\nu}{\rho c} \tag{16}$$

so that

$$F_{\mu\nu} = \frac{m}{e}\left(\frac{a_\mu R_\nu}{\rho} - \frac{a_\nu R_\mu}{\rho}\right)$$

But the actual external field as seen by the particle is an average over all the null directions R_μ/ρ on the backward light cone from the field point x^μ. It is well known that the average of R_μ/ρ is v_μ/c. So

$$\bar{F}_{\mu\nu} = (m/ec)(a_\mu v_\nu - a_\nu v_\mu) \tag{17}$$

The bar indicates the null cone average. But this is just a Fermi–Walker propagator. So the Lorentz equation is seen to lead to an expression for Fermi–Walker transport. A similar propagation equation for accelerating charges was recently derived by Hogan and Ellis (1989).

This type of analysis can be used to compute the observed current of the external field $j_e^\nu = (c/4\pi)\, \partial\bar{F}^{\mu\nu}/\partial x^\mu$. As in (16), $\partial a_\mu/\partial x^\nu = \dot{a}_\mu R_\nu/\rho c$, so that

$$j_e^\nu = \frac{m}{4\pi e}\left(\frac{\dot{a}^\mu R_\mu v^\nu}{\rho c} - \frac{\dot{a}^\nu R_\mu v^\mu}{\rho c}\right)$$

This vanishes either for uniform acceleration or for a null cone average. This is to be expected, since the source of the external field is distant.

The effect of this gauge on the total dynamics of the particle can now be examined. A recent theory of the extended charged particle (Beil, 1989*a*) is used. In this theory the particle extension is in the ρ direction and is described by a shape function $g(\rho)$. The self-potential of the particle is

$$A_s^\mu = \frac{e}{c}\frac{g(\rho)}{\rho} v^\mu$$

This produces the field tensor

$$F_s^{\mu\nu} = \frac{e}{\rho c}\left[(R^\mu a^\nu - R^\nu a^\mu)\frac{g}{\rho c} - (R^\mu v^\nu - R^\nu v^\mu)\left(1 - \frac{a_\alpha R^\alpha}{c^2}\right)\frac{d}{d\rho}\left(\frac{g}{\rho}\right)\right] \tag{18}$$

and the self-current,

$$\begin{aligned} j_s^\mu &= -\frac{c}{4\pi}\frac{\partial F_s^{\mu\nu}}{\partial x^\nu} \\ &= \frac{e}{4\pi\rho}\left\{v^\mu\left[\frac{a_\alpha R^\alpha}{c^2}\left(\frac{1}{\rho}\frac{dg}{d\rho} + \frac{d^2 g}{d\rho^2}\right) - \frac{d^2 g}{d\rho^2}\right] + \frac{a^\mu}{c}\frac{dg}{d\rho}\right. \\ &\quad + \frac{R^\mu}{\rho c}\left[(a_\alpha R^\alpha)\left(\frac{d^2 g}{d\rho^2} - \frac{1}{\rho}\frac{dg}{d\rho}\right) - \frac{\dot{a}_\alpha R^\alpha}{c}\frac{dg}{d\rho}\right. \\ &\quad \left.\left. - \frac{(a_\alpha R^\alpha)^2}{c^2}\left(\frac{d^2 g}{d\rho^2} - \frac{1}{\rho}\frac{dg}{d\rho}\right)\right]\right\} \end{aligned}$$

The total current over a 3-space volume is

$$J_s^\mu = \frac{1}{c}\int j_s^\mu \rho^2\, d\rho\, d\Omega$$

$$= \frac{e}{c}\int_0^\infty \rho\left[-v^\mu \frac{d^2 g}{d\rho^2} + \frac{4}{3}\frac{a^\mu}{c}\frac{dg}{d\rho} - \frac{\rho}{3}\frac{a^\mu}{c}\frac{d^2 g}{d\rho^2} + \frac{\rho}{3}\frac{\dot a^\mu}{c^2}\frac{dg}{d\rho} + \frac{\rho^2}{3}\frac{a^2 v^\mu}{c^4}\frac{d^2 g}{d\rho^2} + \rho\frac{a^2 v^\mu}{c^4}\frac{dg}{d\rho}\right] d\rho$$

This can be simplified by integration by parts and the charge normalization condition,

$$\int_0^\infty \rho \frac{d^2 g}{d\rho^2}\, d\rho = -1$$

to

$$J_s^\mu = \frac{e}{c} v^\mu + 2\frac{e}{c^2} a^\mu \int_0^\infty \rho \frac{dg}{d\rho}\, d\rho + \frac{e}{3c^3}\dot a^\mu \int_0^\infty \rho^2 \frac{dg}{d\rho}\, d\rho \qquad (19)$$

The momentum transfer of the self-field (18) could be computed as in Beil (1989*a*). The results there show the appearance in the equation of motion of a rest mass term, the Larmor term, and additional terms involving the rate of change of the shape function. This is all unaffected by the gauge choice for the external field.

The momentum transfer of interest here is that due to the interaction of the self-field (18) and the external field (17). The equation which results for the rate of change of the interaction four-momentum P_i is

$$\frac{dP_i^\mu}{d\tau} = \frac{e}{c}\bar F_e^{\mu\nu} v_\nu + \frac{e}{3c^3}\bar F_e^{\mu\nu}\dot a_\nu \int_0^\infty \rho^2 \frac{dg}{d\rho}\, d\rho + 2\frac{e}{c^2}\bar F_e^{\mu\nu} a_\nu \int_0^\infty \rho \frac{dg}{d\rho}\, d\rho$$

[see equation (49) of Beil (1989*a*)].

Now, when explicit use is made of (17), we obtain

$$\frac{dP_i^\mu}{d\tau} = ma^\mu - 2\frac{m}{c^3} a^2 v^\mu \int_0^\infty \rho \frac{dg}{d\rho}\, d\rho - \frac{m}{3c^4}(a^2 a^\mu + a^\nu \dot a_\nu v^\mu)\int_0^\infty \rho^2 \frac{dg}{d\rho}\, d\rho$$

With the assumption of uniform acceleration, $\dot{a}^{\mu} = -(a^2/c^2)v^{\mu}$, the interaction energy-momentum is

$$P_i^{\mu} = mv^{\mu} + 2\frac{m}{c}a^{\mu}\int_0^{\infty}\rho\frac{dg}{d\rho}\,d\rho + \frac{m}{3c^2}\dot{a}^{\mu}\int_0^{\infty}\rho^2\frac{dg}{d\rho}\,d\rho$$

But comparison with (19) immediately implies

$$J_s^{\mu} = (e/mc)P_i^{\mu}$$

This shows the connection between the current of the extended particle and the interaction with an external field. It indicates that the natural gauge is consistent with the extended particle model. Obviously, for a point particle $J_s^{\mu} = (e/c)v^{\mu}$, as expected.

REFERENCES

Apsel, D. (1979). *General Relativity and Gravitation*, **10**, 279.
Beil, R. G. (1987). *International Journal of Theoretical Physics*, **26**, 189.
Beil, R. G. (1989*a*). *Foundations of Physics*, **19**, 319.
Beil, R. G. (1989*b*). *International Journal of Theoretical Physics*, **28**, 659.
Dirac, P. A. M. (1951). *Proceedings of the Royal Society A*, **209**, 291.
Dirac, P. A. M. (1952). *Proceedings of the Royal Society A*, **212**, 330.
Ferrari, J. A., Griego, J., and Falco, E. E. (1989). *General Relativity and Gravitation*, **21**, 69.
Fontaine, M., and Amiot, P. (1983). *Annals of Physics*, **147**, 269.
Hogan, P. A., and Ellis, G. F. R. (1989). *Annals of Physics*, **195**, 293.
Kawaguchi, T., and Miron, R. (1989). *Tensor*, **48**, 52.
Konopinski, E. J. (1978). *American Journal of Physics*, **46**, 499.
Leibbrandt, G. (1987). *Reviews of Modern Physics*, **59**, 1067.
Parrott, S. (1987). *Relativistic Electrodynamics and Differential Geometry*, Springer-Verlag, New York, p. 190.
Rohrlich, F. (1965). *Classical Charged Particles*, Addison-Wesley, Reading, Massachusetts, p. 231.
Schweizer, M. A. (1990). *American Journal of Physics*, **58**, 930.
Synge, J. L. (1971). *Relativity, The General Theory*, North-Holland, Amsterdam, p. 383.